
ECE 307 – Techniques for Engineering Decisions

3. Introduction to the Simplex Algorithm

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SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- We examine the solution of

$$\underline{A} \underline{x} = \underline{b}$$

using *Gauss—Jordan* elimination

- We first use a simple example and then generalize to cases of general interest
- Consider the system of two equations in five unknowns:

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

$$S_1 \begin{cases} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 & (i) \\ x_1 - x_2 - x_3 - 3x_4 - x_5 = 4 & (ii) \end{cases}$$

□ For this simple example, the number of unknowns exceeds the number of equations and so the system has multiple solutions; this is the principal reason that the *LP* solution is *nontrivial*

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

□ The *Gauss—Jordan* elimination uses *elementary row*

operations:

- multiplication of any equation by a nonzero constant
- addition to any equation of a nonzero constant multiple of any other equation

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

□ We transform by multiplying equation (i) by -1

and then adding it to equation (ii) so as to zero

out the coefficient of x_1

$$S_2 \left\{ \begin{array}{l} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 \\ x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{array} \right.$$

DEFINITIONS

- ❑ A *basic variable* is a variable x_i that appears with the coefficient 1 in an equation and with the coefficient 0 in all the other equations
- ❑ A variable x_j that is *not* basic is called a *nonbasic variable*
- ❑ In the system S_2 , x_1 appears as a *basic* variable; x_2, x_3, x_4 and x_5 are *nonbasic* variables
- ❑ Basic variables may be generated through the use of *elementary row operations*

DEFINITIONS

- ❑ A *pivot operation* is the set of sequential elementary row operations that reduces a system of linear equations into the form in which a specified variable becomes a *basic variable*
- ❑ A *canonical system* is a set of linear equations obtained through *pivot operations* with the property that the system has the same number of *basic variables* as the number of equations in the set

CANONICAL SYSTEM FORM

- We transform the system S_2 into the canonical form of system S_3 :

$$S_3 \left\{ \begin{array}{l} x_1 - 3x_3 - 2x_4 - 4x_5 = 6 \\ x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{array} \right.$$

- The *basic solution* is obtained from a canonical system with all the nonbasic variables set to 0
- For the example, we set $x_3 = x_4 = x_5 = 0$ and so
 $x_1 = 6$ and $x_2 = 2$

BASIC FEASIBLE SOLUTION

- ❑ A *basic feasible solution* is a basic solution in which the *value of each basic variable is nonnegative*
- ❑ In the example of system S_2 , we may choose any two variables to be basic
- ❑ In general for a system of m equations in n unknowns there are $\binom{n}{m}$ possible combinations of basic variables

BASIC FEASIBLE SOLUTION

❑ As n increases, the number of combinations becomes large, even though it remains finite

❑ For the example, we have

$$\binom{5}{2} = \frac{5!}{3! 2!} = 10$$

combinations of possible choices

THE SIMPLEX SOLUTION METHOD

- ❑ We next use a simple example to construct the *simplex* solution method
- ❑ The *simplex method* is a *systematic and efficient scheme* to examine a *subset* of the basic feasible solutions of the *LP* to hone in on *an* optimal solution
- ❑ We apply the notions introduced in the definitions we introduced above

SIMPLEX METHODOLOGY EXAMPLE

$$\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

s.t.

canonical form

$$\begin{cases} x_1 + 2x_2 + 2x_3 + \mathbf{x}_4 & = 8 & (*) \\ 3x_1 + 4x_2 + x_3 & + \mathbf{x}_5 = 7 & (**) \end{cases}$$

$$x_i \geq 0 \quad i = 1, \dots, 5$$

THE SIMPLEX SOLUTION METHOD

- The *canonical form* of the example allows the determination of a basic feasible solution

$$x_1 = x_2 = x_3 = 0 \quad x_4 = 8, \quad x_5 = 7$$

- The corresponding value of the objective is

$$Z = -8 + 7 = -1$$

- The next step is to **improve** the *basic feasible solution* and we need to **find an adjacent basic feasible solution**

ADJACENT FEASIBLE SOLUTION

□ An *adjacent basic feasible solution* is one which differs from the current basic feasible solution in *exactly one basic variable*

□ Note, we characterize a *basic feasible solution* by the following traits

$$\text{basic variable} \geq 0$$

$$\text{nonbasic variable} = 0$$

ADJACENT FEASIBLE SOLUTION

- ❑ The search for an adjacent basic feasible solution uses the idea to change a *nonbasic* variable into a *basic* variable by increasing its value from 0 to the largest positive value **without any constraint violations**
- ❑ To make the search efficient, we select *the nonbasic* variable that improves the value of Z by the largest amount for the *maximization objective*

ADJACENT FEASIBLE SOLUTION

□ In the example, consider the *nonbasic* variable

x_1 , we leave $x_2 = x_3 = 0$ and examine the

possibility to convert x_1 into a basic variable

□ The variable x_1 enters in both constraints

$$x_1 + x_4 = 8$$

$$3x_1 + x_5 = 7$$

ADJACENT FEASIBLE SOLUTION

- The largest value x_1 may assume without making either x_4 or x_5 negative is

$$\min \left\{ 8, \frac{7}{3} \right\} = \frac{7}{3}$$

- We have the new *basic* variable with the value

$$x_1 = \frac{7}{3} ,$$

and the other *basic* variable is

$$x_4 = \frac{17}{3}$$

ADJACENT FEASIBLE SOLUTION

and the three *nonbasic* variables are set to 0:

$$x_2 = x_3 = 0 \text{ and } x_5 = 0$$

□ Note that we obtain an improvement in Z since its value becomes

$$Z = 5 \cdot \frac{7}{3} - \frac{17}{3} = \frac{18}{3} = 6 > -1$$

□ We next need to put the system of equations into *canonical form*:

SIMPLEX METHODOLOGY EXAMPLE

$$\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

s.t.

non - canonical form for x_1

$$\left\{ \begin{array}{l} x_1 + 2x_2 + 2x_3 + x_4 = 8 \quad (*) \\ 3x_1 + 4x_2 + x_3 + x_5 = 7 \quad (**) \end{array} \right.$$

$$x_i \geq 0 \quad i = 1, \dots, 5$$

ADJACENT FEASIBLE SOLUTION

- multiply equation (**) by $-\frac{1}{3}$ and add to equation (*)

$$\frac{2}{3}x_2 + \frac{5}{3}x_3 + x_4 - \frac{1}{3}x_5 = \frac{17}{3}$$

- multiply equation (**) by $\frac{1}{3}$

$$x_1 + \frac{4}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_5 = \frac{7}{3}$$

THE SIMPLEX SOLUTION METHOD

- We continue this process until the *condition of optimality* is satisfied:
 - in a maximization problem, a *basic feasible solution* is *optimal* if and only if the relative profits of each *nonbasic variable* is ≤ 0
 - in a minimization problem, a basic feasible solution is optimal if and only if the relative costs of each *nonbasic variable* is ≥ 0

THE SIMPLEX SOLUTION METHOD

□ The *relative profits (costs)* are given by the change in

Z corresponding to a unit change in a *nonbasic*

variable

□ We use this fact to select the next *nonbasic variable*

to enter the basis

SIMPLEX ALGORITHM FOR MAXIMIZATION

- Step 1:** start with an *initial basic feasible solution* with all constraint equations in *canonical form*
- Step 2:** check for **optimality condition**: if the relative profits are ≤ 0 for each *nonbasic variable*, then *the basic feasible solution is optimal and stop*; else, go to Step 3

SIMPLEX ALGORITHM FOR MAXIMIZATION

- Step 3:** select a *nonbasic variable* to become the new *basic variable*; check the limits on the *nonbasic variable* – the limiting constraint determines which *basic variable* is replaced by the selected *nonbasic variable*
- Step 4:** construct the *canonical form* for the new set of basic variables through *elementary row operations*; evaluate the *basic feasible solution* and Z and return to Step 2

THE SIMPLEX TABLEAU

- ❑ We use an efficient way to represent **visually** the steps in the simplex method through a sequence of so-called tableaus
- ❑ We illustrate the tableau for the simple example for the initial basic feasible solution

THE SIMPLEX TABLEAU

coefficients of the basic variables in Z

coefficient of x_j in Z

$\underline{c}_B$	c_j	5	2	3	-1	1	<i>constraint constants</i>
<i>basic variables</i>		x_1	x_2	x_3	x_4	x_5	
-1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7

THE SIMPLEX TABLEAU

- The optimality check requires the evaluation of

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \cdot \begin{array}{l} \text{column corresponding} \\ \text{to } x_j \text{ in canonical form} \end{array} \right)$$

- For each *nonbasic variable* x_j , for our example, we have

$$\tilde{c}_1 = 5 - [-1, 1] \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3$$

$$\tilde{c}_2 = 2 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$

$$\tilde{c}_3 = 3 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

THE SIMPLEX TABLEAU

- We interpret each $\tilde{c}_j$ as the change in Z in response to a unit increase in x_j

$\underline{c}_B$	c_j	5	2	3	-1	1	<i>constraint</i>
	<i>basic variables</i>	x_1	x_2	x_3	x_4	x_5	<i>constants</i>
-1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7
$\underline{\tilde{c}}^T$		3	0	4	0	0	$Z = -1$

SIMPLEX TABLEAU

- Note that the optimality test indicates that

$$\tilde{c}_1 = 3 > 0 \quad \text{and} \quad \tilde{c}_3 = 4 > 0$$

and so the *initial basic feasible solution* is not optimal

- Since $\tilde{c}_3 > \tilde{c}_1$, we pick x_3 as the *nonbasic variable* to become a *basic variable*

- We examine the limiting solution for x_3 in the two constraint equations:

THE SIMPLEX TABLEAU

<i>equation</i>	<i>limiting basic variable</i>	<i>upper limit on x_3</i>
1	x_4	$(8/2) = 4$
2	x_5	$(7/1) = 7$

and so the limiting value is

$$\min \{4, 7\} = 4$$

□ We replace the basic variable x_4 by x_3

SIMPLEX METHODOLOGY EXAMPLE

$$\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

s.t.

canonical form in x_4 and x_5

$$\left\{ \begin{array}{lcl} x_1 + 2x_2 + 2x_3 + x_4 & = & 8 \quad (*) \\ 3x_1 + 4x_2 + x_3 + x_5 & = & 7 \quad (**) \end{array} \right.$$

$$x_i \geq 0 \quad i = 1, \dots, 5$$

THE SIMPLEX TABLEAU

□ For the new basic feasible solution, we put the equations into canonical form

○ multiplying (*) by $\frac{1}{2}$ to produce (* †)

○ subtract (* †) from (**) to produce (** †)

$$\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 = 4 \text{ (* †)}$$

$$\frac{5}{2}x_1 + 3x_2 - \frac{1}{2}x_4 + x_5 = 3 \text{ (** †)}$$

□ The adjacent basic feasible solution is

$$x_1 = x_2 = x_4 = 0 \qquad x_3 = 4, \quad x_5 = 3$$

THE SIMPLEX TABLEAU

$\underline{c}_B$	c_j <i>basic variables</i>	5	2	3	-1	1	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
3	x_3	1/2	1	1	1/2		4
1	x_5	5/2	3		-1/2	1	3
$\tilde{\underline{c}}^T$		1	-4	0	-2	0	$Z = 15$

$$x_3 = 4, x_5 = 3$$

THE SIMPLEX TABLEAU

- Since $\tilde{c}_1 > 0$, the basic feasible solution is non-optimal
- We examine how to bring x_1 into the basis

<i>equation</i>	<i>limiting basic variable</i>	<i>upper limit on x_1</i>
$(* \uparrow)$	x_3	$4/(1/2) = 8$
$(* * \uparrow)$	x_5	$3/(5/2) = 6/5$

THE SIMPLEX TABLEAU

□ The variable x_1 enters the basis with the value

$$\min \left\{ 8, \frac{6}{5} \right\} = \frac{6}{5}$$

and x_5 is replaced as a basic variable by x_1

□ We need to put the equations

$$\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 = 4 \quad (*\dagger)$$

$$\frac{5}{2}x_1 + 3x_2 - \frac{1}{2}x_4 + x_5 = 3 \quad (**\dagger)$$

into *canonical form* for the *basic variables* x_3 and x_1

THE SIMPLEX TABLEAU

□ The following elementary row operations are used

○ multiply $(**\dagger)$ by $-1/5$ and add to $(*\dagger)$

$$\frac{2}{5}x_2 + \textcolor{violet}{x}_3 + \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{17}{5}$$

○ multiply $(**\dagger)$ by $2/5$

$$\textcolor{violet}{x}_1 + \frac{6}{5}x_2 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = \frac{6}{5}$$

and construct the corresponding tableau

THE SIMPLEX TABLEAU

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	5	2	3	-1	1	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
3	x_3		2/5	1	3/5	-1/5	17/5
5	x_1	1	6/5		-1/5	2/5	6/5
$\underline{\tilde{c}}^T$		0	-26/5	0	-9/5	-2/5	$Z = 81/5$

$\tilde{c}_j \leq 0$ implies optimality

$\overbrace{16.2} > 15$

SIMPLEX TABLEAU EXAMPLE

$$\max Z = 3x_1 + 2x_2$$

s.t.

$$-x_1 + 2x_2 \leq 4$$

$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

SIMPLEX TABLEAU EXAMPLE

□ We put this problem into standard form:

$$\max Z = 3x_1 + 2x_2$$

s.t.

$$-x_1 + 2x_2 + x_3 = 4$$

$$3x_1 + 2x_2 + x_4 = 14$$

$$x_1 - x_2 + x_5 = 3$$

$$x_1, \dots, x_5 \geq 0$$

canonical form

□ x_3, x_4, x_5 are *fictitious* variables

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	3	2	0	0	0	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
0	x_3	-1	2	1			4
0	x_4	3	2		1		14
0	x_5	1	-1			1	3
$\underline{\tilde{c}}^T$		3	2	0	0	0	$Z = 0$

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \bullet \text{column corresponding to } x_j \right)$$

SIMPLEX TABLEAU EXAMPLE

□ The data in $\tilde{c}^T$ indicate that the highest relative profits correspond to x_1 *and* so we want to make x_1 to become a basic variable

□ To bring x_1 into the basis requires to evaluate

$$\min \left\{ \infty, \frac{14}{3}, 3 \right\} = 3$$

and so x_1 replaces x_5 with the value 3

□ We evaluate the basic variable at the adjacent basic feasible solution and convert into canonical form; the new tableau becomes

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	3	2	0	0	0	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
0	x_3		1	1		1	7
0	x_4		5		1	-3	5
3	x_1	1	-1			1	3
$\underline{\tilde{c}}^T$		0	5	0	0	-3	$Z = 9$

SIMPLEX TABLEAU EXAMPLE

- We reproduce here the calculation of the $\underline{\tilde{c}}^T$ components

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \cdot \text{column corresponding to } x_j \right)$$

for each nonbasic variable x_j

- Note that, by definition, $\tilde{c}_i = 0$ for each basic variable x_i

SIMPLEX TABLEAU EXAMPLE

□ The calculations give

$\tilde{c}_1 = 0$ by definition since x_1 is in the basis

$$\tilde{c}_2 = 2 - [0 \ 0 \ 3] \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = 5$$

← indicates possible improvement

$\tilde{c}_3 = 0$ by definition since x_3 is in the basis

$\tilde{c}_4 = 0$ by definition since x_4 is in the basis

$$\tilde{c}_5 = 0 - [0 \ 0 \ 3] \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = -3$$

SIMPLEX TABLEAU EXAMPLE

- Clearly, the only choice is to get x_2 into the basis and so we need to establish the limiting condition from the three equations by evaluating

$$\min \{7, 1, \infty\} = 1$$

and so x_2 replaces x_4 , which becomes a nonbasic variable

- We need to rewrite the equations into canonical form for x_3 and x_2 and construct the new tableau

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	3	2	0	0	0	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
0	x_3			1	-1/5	8/5	6
2	x_2		1		1/5	-3/5	1
3	x_1	1			1/5	2/5	4
$\underline{\tilde{c}}^T$		0	0	0	-1	0	$Z = 14$

$$\underbrace{\tilde{c}_j \leq 0 \quad \forall j}_{\text{all reduced costs are non-positive}} \Rightarrow \text{optimum}$$

SIMPLEX TABLEAU EXAMPLE

□ An optimum is at the solution of

$$\left. \begin{array}{rcl} x_3 & - & \frac{1}{5}x_4 + \frac{8}{5}x_5 = 6 \\ x_2 & + & \frac{1}{5}x_4 - \frac{2}{5}x_5 = 1 \\ x_1 & + & \frac{1}{5}x_4 + \frac{2}{5}x_5 = 4 \end{array} \right\} \text{canonical form in } x_1, x_2 \text{ and } x_3$$

SIMPLEX TABLEAU EXAMPLE

□ This optimum is given by

$$x_4 = x_5 = 0$$

$$x_3 = 6$$

$$x_2 = 1$$

$$x_1 = 4$$

LINEAR PROGRAMMING EXAMPLE

□ Consider the following *LP*

$$\max Z = 3x_1 + 2x_2$$

s.t.

$$-x_1 + 2x_2 \leq 4$$

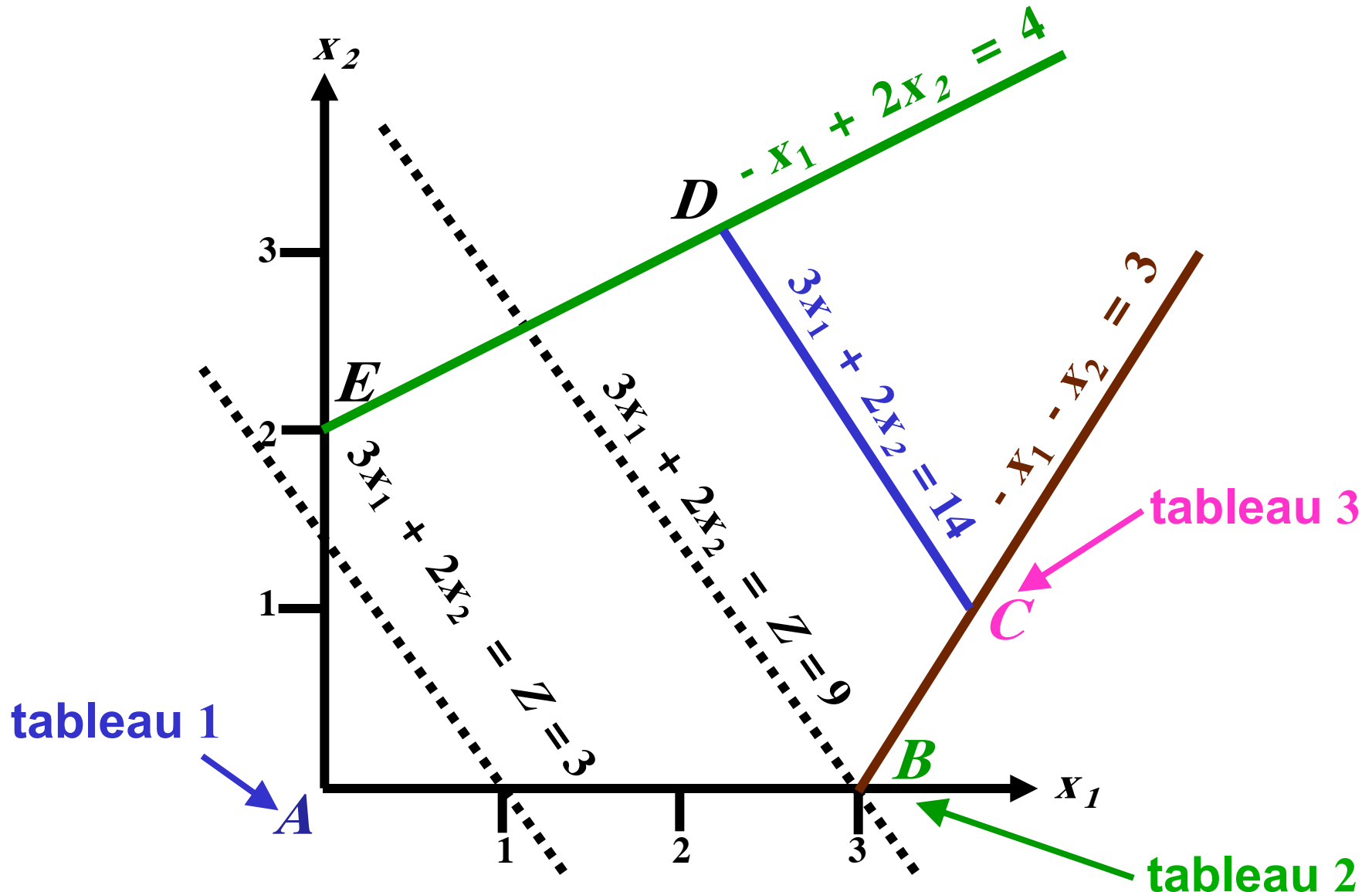
$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

□ The graphical representation corresponds to

LINEAR PROGRAMMING EXAMPLE



LINEAR PROGRAMMING EXAMPLE

- The tableau approach leads to C which is an optimal solution with

$$x_1 = 4, x_2 = 1, x_3 = 6, x_4 = 0, x_5 = 0$$

- Note that any point along CD has $Z = 14$ and as such D is another optimal solution corresponding to an adjacent basic feasible solution
- We may obtain D from C by bringing into the basis the nonbasic variable x_5 in Tableau 3; note that $\tilde{c}_5 = 0$

LINEAR PROGRAMMING EXAMPLE

- We may choose x_5 as a basic variable without affecting Z since the relative profits are 0 ; we compute the limiting value of x_5
- The limit is imposed by x_3 which, consequently, leaves the basis
- The corresponding tableau is:

LINEAR PROGRAMMING EXAMPLE

$\underline{c}_B$	$\begin{matrix} c_j \\ \text{basic} \\ \text{variables} \end{matrix}$	3	2	0	0	0	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	
0	x_3			5/8	-1/8	1	15/4
2	x_2		1	3/8	1/8		13/4
3	x_1	1		-1/4	1/4		5/2
$\underline{\tilde{c}}^T$		0	0	0	-1	0	$Z = 14$

$$\underbrace{\tilde{c}_j \leq 0}_{\forall j}$$

LINEAR PROGRAMMING EXAMPLE

□ The adjacent feasible solution is given by

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{13}{4}, \quad x_3 = x_4 = 0, \quad x_5 = \frac{15}{4}$$

□ Note that at this basic feasible solution,

$$\tilde{c}_j \leq 0 \quad \forall j$$

and so this is also an *optimal solution*

ALTERNATE OPTIMAL SOLUTION

In general, an *alternate optimal solution* is indicated

whenever there exists a *nonbasic variable* x_j with $\tilde{c}_j = 0$

in an optimal tableau; such a situation points to a

non unique optimum for the *LP*

MINIMIZATION *LP*

□ Consider a minimization *LP* with the form given

by

$$\min \quad Z = \sum_{i=1}^n c_i x_i$$

s.t.

$$\underline{A}\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

MINIMIZATION *LP*

□ We replace the optimality check in the simplex scheme by:

if each coefficient $\tilde{c}_j$ is ≥ 0 , stop; else, select

the nonbasic variable with the *most negative*

valued $\tilde{c}$ component to become the *new basic*

variable

MINIMIZATION LP

□ Every minimization LP may be solved as a maximization LP because of equivalence

$$\begin{array}{ll} \min & Z = \underline{c}^T \underline{x} \\ \text{s.t.} & \end{array} \quad \begin{array}{ll} \text{max} & Z' = (-\underline{c}^T) \underline{x} \\ \text{s.t.} & \end{array}$$
$$\begin{array}{ll} \underline{A} \underline{x} = \underline{b} & \underline{A} \underline{x} = \underline{b} \\ \underline{x} \geq \underline{0} & \underline{x} \geq \underline{0} \end{array}$$

with the solutions of Z and Z' related by

$$\min \{ Z \} = - \max \{ Z' \}$$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- ❑ Two variables x_j and x_k are tied in the selection of the *nonbasic variable* to replace a current basic variable when $\tilde{c}_j = \tilde{c}_k$; the choice of the new *nonbasic variable* to enter the basis is *arbitrary*
- ❑ Two or more constraints may give rise to the same *minimum ratio value* in selecting the basic variable to be replaced
- ❑ We consider the example of the following tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	$\begin{matrix} c_j \\ \text{basic} \\ \text{variables} \end{matrix}$	0	0	0	2	0	$3/2$	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_1	1			1	-1	0	2
0	x_2		1		2	0	1	4
0	x_3			1	1	1	1	3
$\underline{\tilde{c}}^T$		0	0	0	2	0	$3/2$	$Z = 0$

candidate for basic variable



COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- in selecting the *nonbasic variable* x_4 to enter the basis, we observe that the first two constraints give the same minimum ratio: this means that when x_4 is first increased to 2, both the basic variables x_1 and x_2 will reduce to 0 even though only one of them can become a *nonbasic variable*
- we *arbitrarily* decide to remove x_1 from the basis to get the new *basic feasible solution*:

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	c_j <i>basic variables</i>	0	0	0	2	0	$3/2$	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4	1			1	-1		2
0	x_2	-2	1			2	1	0
0	x_3	-1		1		1	1	1
$\underline{\tilde{c}}^T$		-2	0	0	0	0	$3/2$	$Z = 4$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

○ in the new basic feasible solution

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 0, \quad \text{and} \quad x_6 = 0 ;$$

we treat x_2 as a *basic variable* whose value is 0,

the same as if it were a *nonbasic variable*

DEGENERACY

- ❑ A *degenerate basic feasible* solution is one which has one or more *basic variables* with the value 0
- ❑ Degeneracy may lead to a number of complications in the simplex approach: an important implication is a minimum ratio of 0 , so that no new *nonbasic variable* may be included in the basis and therefore the basis remains unchanged
- ❑ We consider the following example tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	0	0	0	2	0	$3/2$	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4		$1/2$		1		$1/2$	2
0	x_5	-1	$1/2$			1	$1/2$	0
0	x_3	1	-1	1			0	1
$\underline{\tilde{c}}^T$		0	-1	0	0	0	$1/2$	$Z = 4$

DEGENERACY

the logical choice being the *nonbasic variable* x_6 to enter the basis; this leads to finding the limiting constraint from two equations

$$\frac{1}{2}x_6 = 2 - x_4$$

$$\frac{1}{2}x_6 = 0 - x_5$$

and no constraint in the third equation; thus

$$x_6 = \min\{4, 0, \infty\}$$

DEGENERACY

- ❑ Degeneracy may result in the construction of new tableaus **without improvement in the objective function value**, thereby reducing the efficiency of the computations: theoretically, an infinite loop, the so-called *cycling*, is possible
- ❑ Whenever ties occur in the minimum ratio rule, an *arbitrary decision* is made regarding which *basic variable* is replaced, ignoring the theoretical consequences of degeneracy and cycling

MINIMUM RATIO RULE COMPLICATIONS

- ❑ The minimum ratio rule may not be able to determine the basic variable to be replaced: this is the case when all equations lead to ∞ as the limit
- ❑ Consider the example and corresponding tableau

$$\max \quad Z = 2x_1 + 3x_2$$

s.t.

$$x_1 - x_2 + x_3 = 2$$

$$-3x_1 + x_2 + x_4 = 4$$

$$x_i \geq 0, \quad i = 1, \dots, 4$$

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	c_j <i>basic variables</i>	2	3	0	0	<i>constraint constants</i>
		x_1	x_2	x_3	x_4	
0	x_3	1	-1	1		2
0	x_4	-3	1		1	4
$\underline{\tilde{c}}^T$		2	3	0	0	$Z = 0$

□ The *nonbasic variable* x_2 enters the basis to replace x_4 and the new tableau is

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	$\begin{array}{c} c_j \\ \text{basic} \\ \text{variables} \end{array}$	2	3	0	0	<i>constraint</i>
		x_1	x_2	x_3	x_4	<i>constants</i>
0	x_3	-2		1	1	6
3	x_2	-3	1		1	4
$\underline{\tilde{c}}^T$		11	0	0	-3	$Z = 12$

❑ We select x_1 to enter the basis but we are unable to get limiting constraints from the two equations

MINIMUM RATIO RULE COMPLICATIONS

$$-2x_1 + x_3 = 6 \qquad x_1 = \frac{1}{2}x_3 - 3$$

$$-3x_1 + x_2 = 4 \qquad x_1 = \frac{1}{3}x_2 - \frac{4}{3}$$

□ In fact, as x_1 increases so do x_2 and x_3 and Z

and therefore, the solution is *unbounded*

□ The failure of the minimum ratio rule to result in a

bound at any simplex tableau implies that the

problem has an *unbounded solution*